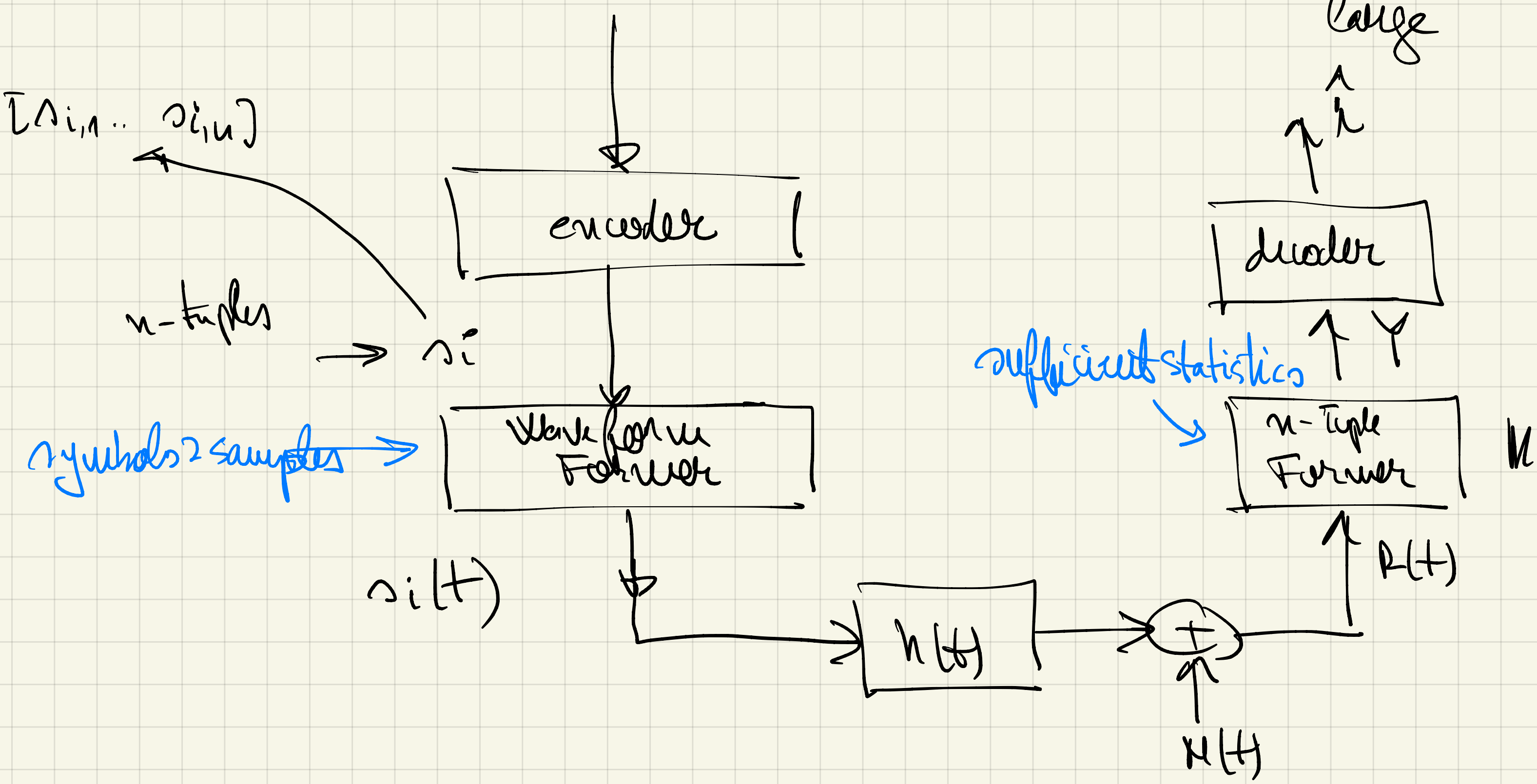
Goal of today's lecture:

Implement a basic point-to-point communication system, for bandlimited Gaussian channels (baseband).

Brief review of PDC (Principles of Digital Communication)

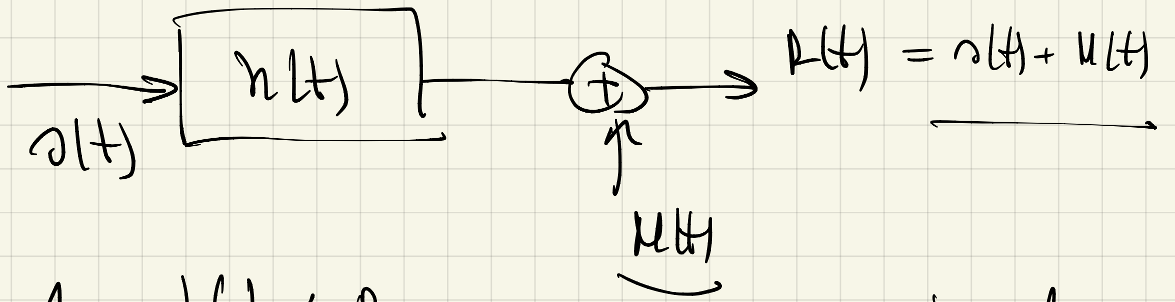
messages: $i \in \{0 \dots M-1\}$ M: finite, possibly very large



Topics:

- Notion of orthonormal expansion
- Nyquist criterion: help construct the orthonormal basis
- Notion of projection: sufficient statistics
- Review of the sampling theorem: we see it as an orthonormal expansion.

The channel



$$h(f) = \begin{cases} 1 & |f| \le B \\ 0 & \text{otherwise} \end{cases}$$

Gaussian noise of power spectral density (PSD) $\frac{N_0}{2}$ per dimension

$s_F(t)$ has support in $[-B, B]$

Digital communications:

Characterized by transmitting a very large set of messages (M) (but finite)

Let $s_1(t) \dots s_M(t)$ be the associated signals, which we assume to be finite norm ($\mathcal{L}_2$).

$s_1(t) \dots s_M(t)$ span an inner-product space, W.

Let $\varphi_1(t) \dots \varphi_n(t)$ be a basis for W.

$$s_i(t) = \sum_{k=1}^{n} s_{i,k} \varphi_k(t)$$

complex symbols to be transmitted

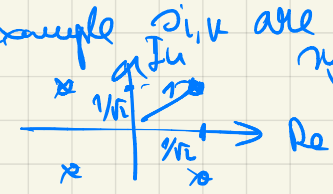
In practice: we do the other way around: we start with $s_1 \dots s_M$ $\quad s_1 = (s_{11} \dots s_{1n})$

and $\varphi_1(t) \dots \varphi_n(t)$

For example $s_{i,k}$ are QPSK symbols

4-QAM: 4 complex symbols "pack" 2 bits

00, 01, 10, 11 $\longleftrightarrow$



Receiver $\quad R(t) = s(t) + N(t)$

Project $R(t)$ onto W $(\varphi_1(t) \dots \varphi_n(t))$

let y be the result

$$y(t) = \sum_{k=1}^{n} y_k \varphi_k(t)$$

$$y_k = \langle R(t), \varphi_k(t) \rangle$$

we get $y = (y_1 \dots y_n) \leftarrow$ values

$Y = (Y_1 \dots Y_n) \leftarrow$ random variables

↳ sufficient statistic

Receiver: minimize the error probability

Decide on the signal which was transmitted by
(symbols)

minimizing $\| y - s_i \|$ $\quad i = 1 \dots n$

$(y_1 \dots y_n)$ $\quad (s_{i1} \dots s_{in})$

Orthonormal basis:

We would like an orthonormal basis $\varphi_1(t) \dots \varphi_k(t)$ such that

$$\varphi_k(t) = \varphi(t - kT) \qquad T: \text{symbol period}$$

$k = 0 \dots n-1$

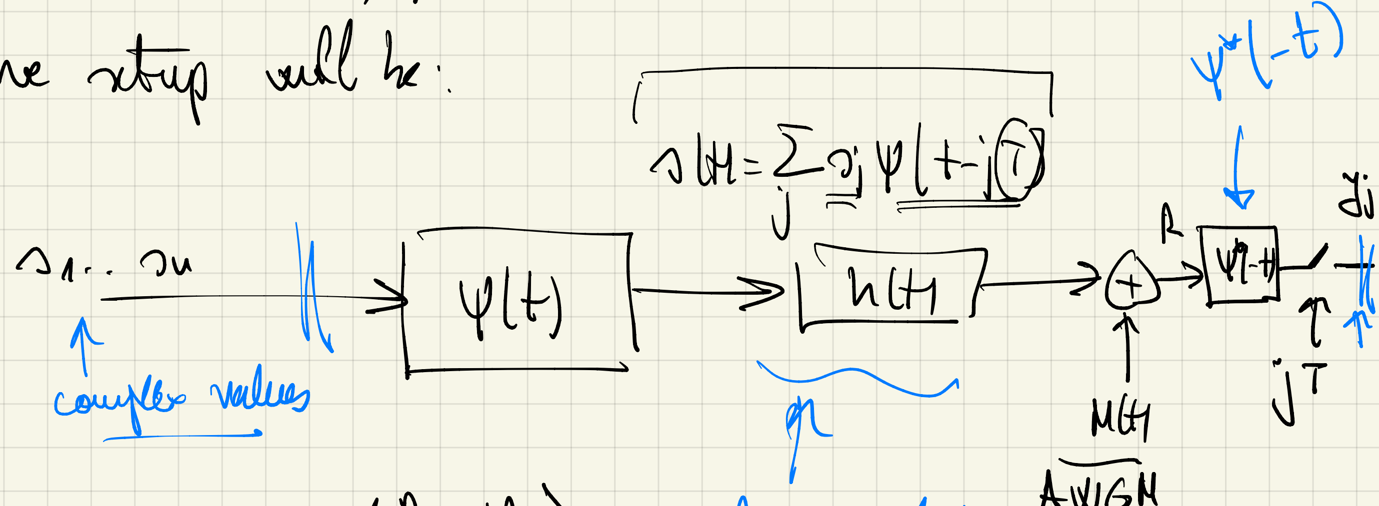
If we choose $\varphi_k(t)$ as above, at the receiver we get y_k by filtering with $\varphi_{MF}(t) = \varphi^*(-t)$ and sampling the output at $t = kT$.
matched filter

$$\underline{y_k} = r(t) * \underbrace{\psi_{MF}(t)}\Big|_{t=kT} = \int r(\alpha)\, \psi_{MF}(t-\alpha)\, d\alpha \Big|_{t=kT} =$$

$$= \int r(\alpha)\, \psi^*(\alpha - t)\, d\alpha \Big|_{t=kT} = \int r(\alpha)\, \underbrace{\psi^*(\alpha - kT)}_{\psi_k^*(\alpha)}\, d\alpha =$$

$$= \langle r(t), \psi_k(t) \rangle$$

The setup will be:



$$\boxed{y_j = \langle R, \psi_j \rangle}$$

Equivalent discrete-time channel:

$$\underline{s_i} \longrightarrow \oplus \longrightarrow y_i = s_i + n_i$$

$\underline{n_i} \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ per complex dimension

How do we choose $\psi_k(t)$?

$$\psi_k(t) \perp \psi_j(t) \quad : \quad \langle \psi_k(t), \psi_j(t) \rangle = \delta_{k-j}$$

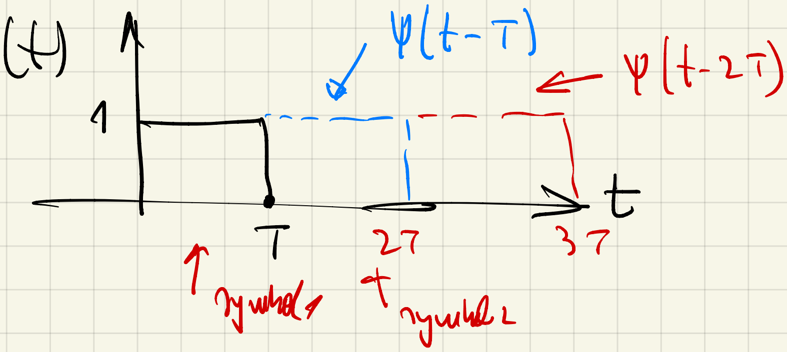
$$\delta_{k-j} = \begin{cases} 1 & \text{when } k=j \\ 0 & \text{otherwise} \end{cases}$$

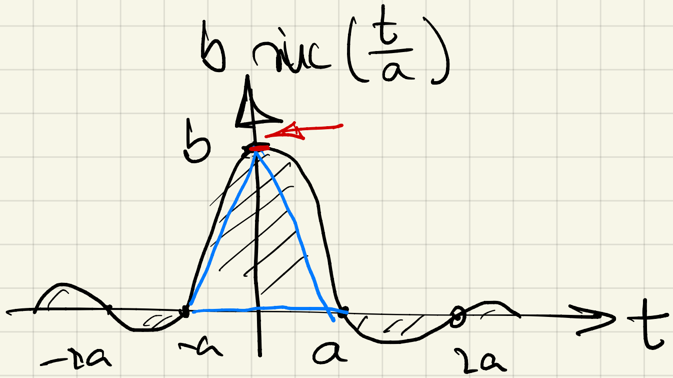
$$\boxed{\psi_k(t) = \psi(t - kT)}$$

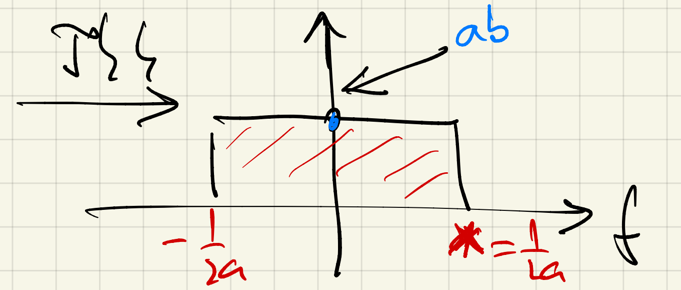
$j=0$

$$\int \psi(t-kT)\, \psi^*(t)\, dt = \delta_k \quad \begin{cases} 1 & \text{for } k=0 \\ 0, & \text{otherwise} \end{cases}$$

Simplest choice: $\psi(t)$







$a\cancel{b}\cdot 2x = \cancel{b} \Rightarrow x = \frac{1}{2a}$

How to choose $\psi(t)$: we would like something that respects:

$$\int \psi(t-kT)\,\psi^*(t)\,dt = \delta_k \qquad (1)$$

We want the frequency equivalent of the previous condition.

Define a periodic function:

$$g(f) = \sum_{k\in\mathbb{Z}} \Psi_F\left(f-\frac{k}{T}\right)\Psi_F^*\left(f-\frac{k}{T}\right) = \sum_{k\in\mathbb{Z}} \left|\Psi_F\left(f-\frac{k}{T}\right)\right|^2$$

This is periodic of period $\frac{1}{T}$.

Fourier series: coefficients

$$\frac{k}{1/T}$$

$$A_k = \frac{1}{1/T}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} g(f)\, e^{-j2\pi kTf}\, df$$

Parseval for (1):

$$\delta_k = \int \Psi_F(f)\, e^{-j2\pi f kT}\, \Psi_F^*(f)\, df =$$

$$= \int |\psi_F(f)|^2 e^{-j2\pi f k T} df =$$

$$= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} g(f)\, e^{-j2\pi f k T} df = \frac{A_k}{T}$$

$$d_k = \frac{A_k}{T} \Rightarrow \begin{array}{l} A_0 = 1 \\ A_{\pm 1}, \ldots \text{ etc} = 0 \end{array}$$

$$d_0 = 1 = \frac{A_0}{T} \Rightarrow \underline{A_0 = T}$$

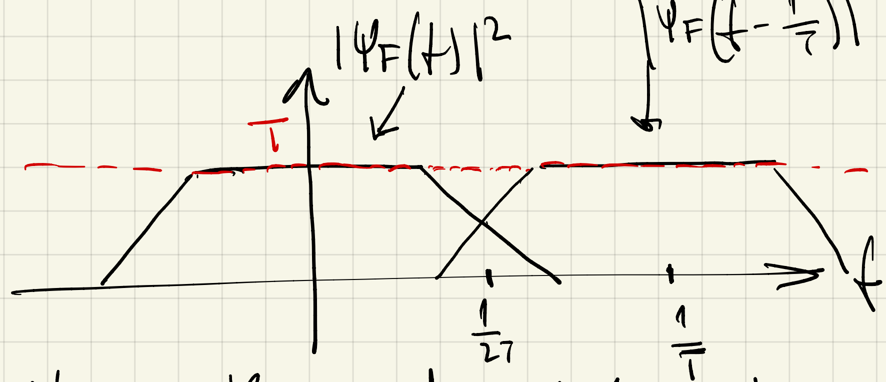
$$\underline{g(f)} = \sum_{k \in \mathbb{Z}} A_k e^{j2\pi f k T} = A_0 = T$$

<u>Theorem (Nyquist)</u>

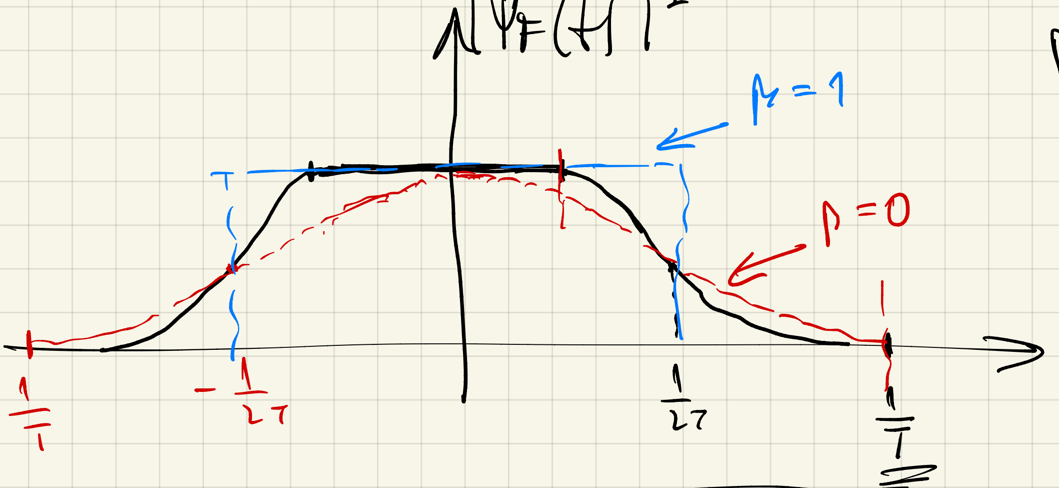
Let $\psi(t)$ be a $\mathcal{L}_2$ function. The ~~set~~ $\psi(t-kT)\big|_{k\in\mathbb{Z}}$ consist of an orthonormal basis ~~if~~ and only if

$$\underset{\substack{\uparrow}}{\underline{\text{l.i.m}}} \sum_{k=-\infty}^{\infty} \left| \psi_F\left(f - \frac{k}{T}\right)\right|^2 = T \quad \text{for } f \in \mathbb{R}$$

limit in mean square



Popular choice for $\psi(t)$ is the root-raised-cosine (RRC) pulse



β: roll-off factor

$\beta = 0$

$\beta = 1$

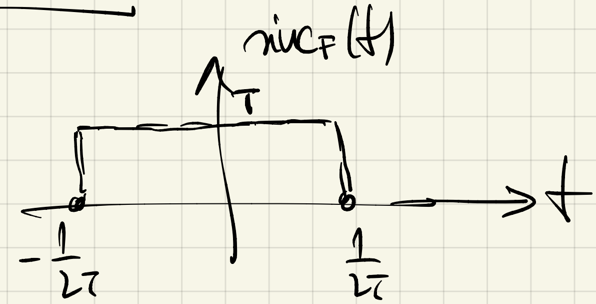
Sampling Theorem

Let $s(t)$ be an $\mathcal{L}_2$ norm function, and $s_F(f)$ having support in $[-B, B]$. (Baseband signal).

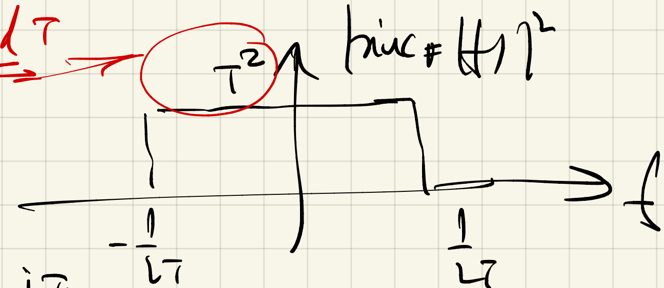
$$s(t) = \sum_i s(iT)\,\mathrm{sinc}\left(\frac{t-iT}{T}\right)$$

for any $\boxed{T \le \frac{1}{2B}}$

$$\mathcal{F}\left\{\mathrm{sinc}\left(\frac{t}{T}\right)\right\} \longrightarrow$$



We need T



Define $\psi(t) = \frac{1}{\sqrt{T}}\,\mathrm{sinc}\left(\frac{t-iT}{T}\right)$

$$\Rightarrow \boxed{s(t) = \sum_i s_i\,\psi_i(t)}$$ where $s_i = \sqrt{T}\,s(iT)$

For the assignment:

we need to compute $s(t) = \sum s_i \, \psi(\underbrace{t - iT})$

(the samples of)

$\uparrow$ QAM symbols for example

$s[k] = \sqrt{T_s}\, s(kT_s)$

$\psi[k] = \sqrt{T_s}\, \psi(kT_s)$

$T = \underline{\underline{M}}\, T_s \leftarrow$ sampling time.

or symbol period

We have M samples in a symbol duration

$$s[n] \doteq s(nT_s) = \sum_k s_k \, \psi(\underbrace{nT_s}_{t|_{t=nT_s}} - k\underbrace{MT_s}_{T})$$

$$= \sum_k s_k \, \psi((n-kM)T_s)$$

$$= \sum_k s_k \, \psi[n-kM] \leftarrow s_0\psi[n] + s_1\psi[n-M]$$

$$= \sum_k \hat{s}_k \, \psi[n-k] \qquad \hat{s}_k = \begin{cases} s_k & \text{if } k \text{ is a multiple of } M \\ 0 & \text{otherwise} \end{cases}$$

$$= (\hat{s}_k * \psi)[n]$$

We introduce $M-1$ zeros in between ~~the~~ any two consecutive values of $s = (s_1 \dots s_n \dots)$